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DESIGN OF LOW SENSITIVITY SAMPLED-DATA CONTROL SYSTEMS

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ABSTRACT

A frequency domain technique is developed for designing the minimum sensitivity deadbeat sampled-data control systems where plant variations are encountered. This is done by modifying a previously proposed design technique for minimal sensitivity systems. The definitions of minimal sensitivity, minimum sensitivity, and deadbeat property are defined. The present method is to reduce the magnitude of the output variation, due to plant variations, by using two controllers. Two examples are given to demonstrate the effectiveness of the method. In general, the magnitude of the output variation for the minimum sensitivity system is smaller than that for the minimal sensitivity system. But both systems are less sensitive to plant variation than the single controller system.

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INTRODUCTION

In a recent paper, a frequency domain method of synthesizing the "minimal sensitivity" sampled-data control systems, with the deadbeat property, has been proposed.(1) The essence of the method was to design two controllers, D_1 and D_2, Figure 1, in such a manner that the variation of the system output, due to the variation of the plant, subsides in the smallest number of sampling periods.

There are cases where the rapidness of subsidence of the variation of the output response is of secondary importance; but, rather, the magnitude of the variation is of primary concern. This paper presents a modification of a previous method, in that a design procedure is developed which minimizes the magnitude of the output variation caused by plant variation. It will be shown that this can be done by trading the subsidence speed for low magnitude of output variation. The design procedure is simple, and two examples will be given to demonstrate the effectiveness of the method.

REVIEW

Let C(z) be the z-transform of the system output. The differential variation of the system output due to a differential variation of the plant parameters is denoted by dC(z). Since dC(z) is a polynomial in z^{-1}, it will be called the "sensitivity polynomial." A "minimal sensitivity" system is defined as a system having the shortest

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sensitivity polynomial. The property of a zero-percent settling within a finite time interval will be called the "deadbeat" property; and a system whose output response possesses this property will be called a deadbeat system.

Referring to Figure 1, the control ratio of the system is

$$H(z) = \frac{C(z)}{R(z)} = \frac{D_1(z)G(z)}{1 + D_1(z)D_2(z)G(z)} \quad . \tag{1}$$

The differential change of the output response is related to the differential change of the plant by

$$dC(z) = R(z)H(z) \left[1 - D_2(z)H(z) \right] \frac{dG(z)}{G(z)} \tag{2}$$

which is termed the "sensitivity polynomial."

In general, the normalized plant variation and the reference input have the forms

$$\frac{dG(z)}{G(z)} = \frac{Q(z)}{P(z)} \tag{3}$$

and

$$R(z) = \frac{A(z)}{(1 - z^{-1})^m} \quad , \tag{4}$$

respectively, where Q(z), P(z), and A(z) are finite polynomials in z^{-1} and A(z) has no roots at z = 1. For a deadbeat system, the control ratio has the form

$$H(z) = z^{-\delta} F(z) \tag{5}$$

where $z^{-\delta}$ is the transport lag of the plant and F(z) is a finite polynomial in z^{-1} with a constant term. Let U(z) and V(z) be the numerator and the denominator polynomial of $D_2(z)$, respectively; i.e.

$$D_2(z) = \frac{U(z)}{V(z)} \quad . \tag{6}$$

Using Eqs. (3), (4), (5), and (6), Eq. (2) can be expressed as

$$dC(z) = \frac{A(z)}{(1 - z^{-1})^m} H(z) \left[1 - \frac{U(z) z^{-\delta} F(z)}{V(z)} \right] \frac{Q(z)}{P(z)} \quad . \tag{7}$$

Let

$$\frac{N(z)}{M(z)} = \frac{A(z)H(z)}{(1 - z^{-1})^m P(z)} \tag{8}$$

where M(z) and N(z) are finite polynomials in z^{-1} and are relatively prime. Then Eq. (7) becomes

$$dC(z) = \frac{N(z)}{M(z)} \left[1 - \frac{U(z) z^{-\delta} F(z)}{V(z)} \right] Q(z) \quad . \tag{9}$$

Minimal sensitivity is obtained when

$$V(z) = F(z) \tag{10}$$

and when U(z) is so chosen that

$$S(z) = \frac{1 - U(z) z^{-\delta}}{M(z)} \tag{11}$$

is the shortest polynomial in z^{-1}. Therefore, the minimal sensitivity polynomial is

$$dC(z) = N(z)S(z)Q(z) \quad . \tag{12}$$

The controller $D_2(z)$ is determined once V(z) and U(z) are determined, and the controller $D_1(z)$ is given by

$$D_1(z) = \frac{H(z)}{G(z) \left[1 - D_2(z)H(z) \right]} \quad . \tag{13}$$

Note that, for a given plant G(z) and input R(z), the above procedure yields a unique

$D_1(z)$ and $D_2(z)$. Details of the development, the discussion, and the design procedure for minimal sensitivity deadbeat sampled-data control systems may be found in Reference 1.

DEVELOPMENT

While the minimal sensitivity design tends to recover the desired output response from the varied output in the least number of sampling periods, it does not reduce the magnitude of the output variation, $dC(z)$. There are practical control applications where a small magnitude of output variation is of primary concern while the speed of recovery is less important.

The magnitude of the output variation can be reduced at the expense of a slower subsidence speed. This is the main theme of this paper and is developed as follows.

Suppose that the shortest polynomial $S(z)$ in Eq. (11) has been found by proper choice of $U(z)$. Consider the equation

$$\frac{1 - U^*(z) z^{-\delta}}{M(z)} = S(z)B(z) \tag{14}$$

where $U^*(z) \neq U(z)$, and $U^*(z)$ is a new numerator for $D_2(z)$. $B(z)$ has the form

$$B(z) = 1 + \sum_{i=1}^{k} b_i \, z^{-i} \quad . \tag{15}$$

The constants b_i in the last equation will be determined. Substituting (14) and (10) into (9), the new sensitivity polynomial is then

$$dC(z) = N(z)S(z)B(z)Q(z) \tag{16}$$

which is longer than Eq. (12) by k sampling periods by virtue of the factor $B(z)$. Expanding Eq. (16), the polynomial of the output variation has the form

$$dC(z) = \sum_{j=1}^{q} \sigma_j \, z^{-j} \tag{17}$$

where the coefficients, σ_j, are linear functions of b_1, b_2, ---, b_k, and q is a finite positive integer depending on the lengths of $N(z)$, $S(z)$, $B(z)$, and $Q(z)$. The sum of the squares of the output variation is given by

$$J = \sum_{j=0}^{q} \left[dc(jT) \right]^2 \tag{18}$$

where T is the sampling period. Hence J is a quadric in b_i, i = 1, ---,k.

The magnitude of the output variation is indirectly suppressed by minimizing J with respect to all b_i. By setting

$$\frac{\partial J}{\partial b_i} = \sum_{j=0}^{q} \frac{\partial}{\partial b_i} \left[dc(jT) \right]^2 = 0, \quad i = 1, ---,k, \tag{19}$$

then, k linear equations are obtained, which can be solved for k unknowns b_i, b_2,---,b_k. Thus, B(z) is uniquely determined. The new numerator of $D_2(z)$ is obtained by using Eq. (14). That is,

$$U^*(z) = z^{\delta} \left[1 - S(z)B(z)M(z) \right] \quad . \tag{20}$$

By using different values of k, different amounts of trade-offs are made between the subsidence speed and the magnitude of the output variation. That is, the larger the k, the smaller the magnitude, and the slower the subsidence speed. This design technique will be called the "minimum sensitivity" design with k "trade-off periods."

DESIGN PROCEDURE

A complete procedure, for designing the minimum sensitivity deadbeat sampled-data control system, is outlined in this section. The procedure for obtaining a deadbeat control ratio is well known; however, it will be included in the outline for completeness. [2,3]

Given the plant

$$G(z) = \frac{Kz^{-\delta} \prod_j (1 - z_j z^{-1})^{m_j}}{\prod_i (1 - p_i z^{-1})^{n_i}} \tag{21}$$

where K, $z^{-\delta}$, z_j, p_i, m_j, and n_i are gain, transport lag, zeros, poles, zero order, and pole order, respectively, the normalized plant variation is given by

$$\frac{dG(z)}{G(z)} = \frac{Q(z)}{D(z)} = \frac{dK}{K} + \sum_i \frac{n_i z^{-1} dp_i}{1 - p_i z^{-1}} - \sum_j \frac{m_j z^{-1} dz_j}{1 - z_j z^{-1}} \quad . \tag{22}$$

For the specified type of input, Eq. (4), the design procedure is as follows:

Step I. Let

$$H(z) = z^{-\delta} \prod_j (1 - z_j z^{-1})^{m_j} \sum_{i=0}^{m-1} a_i z^{-i} \tag{23}$$

and determine a_i from the condition

$$\left.\begin{aligned} &H(1) = 1 \\ &H^{(\alpha)}(1) = 0 \qquad \alpha = 1, ---, m-1 \end{aligned}\right\} , \tag{24}$$

where $H^{(\alpha)}(z)$ is the α-th derivatives of H(z) with respect to z^{-1}.

Step II. Determine $\frac{dG(z)}{G(z)} = \frac{Q(z)}{p(z)}$ as given by Eq. (22) and determine M(z) and N(z) using Eq. (8).

Step III. Let

$$V(z) = F(z) = \prod_j (1 - z_j z^{-1})^{m_j} \sum_{i=1}^{m-1} a_i z^{-i} \tag{25}$$

and obtain the shortest polynomial S(z) by choosing a proper U(z) for Eq. (11).

Step IV. For a minimum sensitivity with k trade-off periods substitute

$$B(z) = 1 + \sum_{i=1}^{k} b_i \; z^{-i} \tag{26}$$

into

$$dC(z) = N(z) \; S(z)B(z)Q(z) \tag{27}$$

and expand the latter into a polynomial in z^{-1}. Now, one can easily write

$$J = \sum_{j=0}^{q} \left[dc(jT) \right]^2 \quad , \tag{28}$$

where q is the order of dC(z), and obtain B(z) by solving for $b_1, b_2, ---, b_k$ from the set of k linear algebraic equations

$$\frac{\partial J}{\partial b_i} = 0, \quad i = 1, ---, k \quad . \tag{29}$$

Step V. The two digital controllers are given by

$$D_2(z) = \frac{U^*(z)}{V(z)} \quad , \tag{30}$$

where

$$U^*(z) = z^{\delta} \left[1 - S(z)B(z)M(z) \right] \quad , \tag{31}$$

and

$$D_1(z) = \frac{H(z)}{G(z)\left[1 - D_2(z)H(z) \right]} \quad . \tag{32}$$

EXAMPLES

Two examples are given to illustrate the design procedure and to demonstrate the effectiveness of the method.

I. Example 1

Given the plant

$$G(z) = \frac{0.3679z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})(1 - 0.3689z^{-1})} \quad ,$$

design the minimum sensitivity deadbeat system for a step input, where the plant gain is the varying parameter.

Step I. Let

$$H(z) = z^{-1}(1 + 0.7183z^{-1})a_o \quad .$$

Letting H(1) = 1, gives $a_o = 0.5820$. Hence

$$H(z) = 0.582z^{-1}(1 + 0.7183z^{-1})$$

and

$$E(z) = 1 + 0.418z^{-1}) \quad .$$

Step II. Using Eq. (22),

$$\frac{dG}{G} = \frac{dK}{K} \quad \text{, a constant} \quad .$$

From Eq. (8),

$$\frac{N(z)}{M(z)} = \frac{0.582z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})} \quad .$$

Step III. Let

$$V(z) = F(z) = 0.582(1 + 0.7183z^{-1}) \quad .$$

By choosing U(z) = 1, gives

$$S(z) = \frac{1 - U(z)z^{-1}}{(1 - z^{-1})} = 1 \quad ,$$

the shortest S(z).

Step IV. For a minimum sensitivity with one trade-off period, k = 1. From Eq. (27)

$$dC(z) = 0.582z^{-1}(1 + 0.7183z^{-1})(1 + b_i z^{-1}) \frac{dK}{K} \quad .$$

By setting

$$\frac{\alpha J}{\alpha b_1} = \sum_{j=1}^{3} \left[dC(jT) \right]^2 = 0$$

and solving for b_1, gives

$$b_1 = -0.4738 \quad .$$

Step V. Using Eqs. (30), (31), and (32),

$$D_1(z) = \frac{1.582(1 - 0.3679z^{-1})}{(1 - 0.4738z^{-1})}$$

and

$$D_2(z) = \frac{2.5323(1 - 0.3215z^{-1})}{(1 + 0.7183z^{-1})} \quad .$$

The design for two trade-off periods has also been done. It would be interesting to compare the performance of the two-controller minimum sensitivity design to that of minimal sensitivity design, and also to that of the one-controller design, Figure 2. The design of the one-controller system can be found from Reference 6. Under the nominal condition, all three systems have identical step response. Table I contains the comparison of the four systems under the condition of 20 percent gain variation. Figure 3 shows the system errors for all four systems under the nominal and the varied conditions. The trade-off effect between the minimum sensitivity and minimal sensitivity designs is evident from this figure.

EXAMPLE 2

In a practical system, the change of a single plant parameter may cause a simultaneous change of gain, poles, and zeros of the plant. Consider a separately excited dc motor driving a load. The transfer function between the applied armature voltage V and the motor shaft position θ is

$$\frac{\theta}{V} = \frac{\dfrac{K_T}{J\ R_a}}{s\left[s + \dfrac{1}{J}\left(B + \dfrac{K_T K_e}{R_a}\right)\right]}$$

where

$$J = 443.0 \text{ slug-ft}^2, \text{ armature-load inertia },$$

$$B = 160 \text{ lb-ft/rad-sec, friction coefficient },$$

$$K_T = 28.2 \text{ lb-ft/amp, torque constant },$$

$$K_e = 4.0 \text{ volt/rad/sec, emf constant },$$

and

$$R_a = 0.1 \text{ ohm, armature resistance} .$$

The motor is preceded by a zero-order hold. Design a deadbeat sampled-data control system with minimal sensitivity in response to step inputs. The sampling period is one second and the varying parameter is the armature circuit resistance, R_a. For example, a 20 percent increase of R_a causes an 18.2 percent decrease in gain, a 2.8 percent decrease in zero and a 10.6 percent decrease in a pole.

Under the nominal condition the following Equations are obtained:

$$G(z) = \frac{0.1092z^{-1}(1 + 0.2639z^{-1})}{(1 - z^{-1})(1 - 0.0104z^{-1})} ,$$

$$H(z) = 0.7912z^{-1}(1 + 0.2639z^{-1}) ,$$

and

$$E(z) = 1 + 0.2088z^{-1} .$$

The results of the two-controller minimum sensitivity design, the two-controller minimal sensitivity design, and the one-controller design are listed in Table II. Figure 4 shows the system errors for all three systems, under the nominal and the varied conditions.

CONCLUSION

By modifying a previously proposed minimal sensitivity design technique, a frequency domain method has been developed for designing the minimum sensitivity deadbeat sampled-data control systems where plant variations are encountered. The systems are designed to reduce the magnitude of output variations due to plant variations by using a two-controller configuration. Two examples are given to demonstrate the effectiveness of the method.

In general, if the plant variation is not too severe, the magnitude of the output variation for the minimum sensitivity system is smaller than that for the minimal sensitivity system. But both types of systems are less sensitive to the plant variation than the single-controller system. The method seems attractive due to its simplicity and effectiveness.

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TABLE I COMPARISON FOR EXAMPLE I

The nominal plant:

$$G(z) = \frac{0.3679z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})(1 - 0.36893^{-1})}$$

Two-controller minimum sensitivity design:

For k = 1, $D_1(z) = \frac{1.582(1 - 0.3679z^{-1})}{(1 - 0.4738z^{-1})}$, $D_2(z) = \frac{2.5323(1 - 0.3215z^{-1})}{(1 + 0.7183z^{-1})}$.

For k = 2, $D_1(z) = \frac{1.582(1 - 0.3679z^{-1})}{1 - 0.6112z^{-1} + 0.2896z^{-2}}$, $D_2(z) = \frac{2.7684(1 - 0.5591z^{-1} + 0.1797z^{-2})}{(1 + 0.7183z^{-1})}$.

Two-controller minimal sensitivity design:

$D_1(z) = 1.582(1 - 0.3679z^{-1})$, $D_2(z) = \frac{1.7183}{1 + 0.7183z^{-1}}$.

One-controller design:

$$D(z) = \frac{1.582(1 - 0.3679z^{-1})}{1 + 0.418z^{-1}} .$$

For a 20% variation of plant gain

The varied plant:

$$G_v(z) = \frac{0.4415z^{-1}(1 + 0.7183z^{-1})}{(1 - z^{-1})(1 - 0.3679z^{-1})} .$$

Two-controller minimum sensitivity system:

For k = 1, $H_v(z) = \frac{0.6985z^{-1}(1 + 0.7183z^{-1})}{1 + 0.2950z^{-1} - 0.0949z^{-2}}$

$$E_v(z) = 1 + 0.3015z^{-1} + 0.006z^{-2} + 0.0268z^{-3} - 0.0073z^{-4} + 0.0047z^{-5} + \cdots$$

For k = 2, $H_v(z) = \frac{0.6985z^{-1}(1 + 0.7183z^{-1})}{1 + 0.3225z^{-1} - 0.1803z^{-2} + 0.0579z^{-3}}$

$$E_v(z) = 1 + 0.3015z^{-1} + 0.0251z^{-1} - 0.0117z^{-3} - 0.0092z^{-4} - 0.0006z^{-5} + \cdots$$

Two-controller minimal sensitivity system:

$$H_v(z) = \frac{0.6985z^{-1}(1 + 0.7183z^{-1})}{1 + 0.2z^{-1}}$$

$$E_v(z) = 1 + 0.3015z^{-1} - 0.0605z^{-2} + 0.0119z^{-3} - 0.0026z^{-4} + 0.0003z^{-5} + \cdots$$

One-controller system:

$$H_v(z) = \frac{0.69853^{-1}(1 + 0.7183z^{-1})}{1 + 0.1165z^{-1} + 0.084z^{-2}}$$

$$E_v(z) = 1 + 0.3015z^{-1} - 0.1188z^{-2} + 0.0114z^{-3} + 0.0112z^{-4} - 0.0003z^{-5} + 0.0009z^{-6} + 0.0001z^{-7} + \cdots$$

TABLE II. COMPARISON FOR EXAMPLE 2.

The nominal condition

Plant: $$G(z) = \frac{0.1092z^{-1}(1 + 0.2639z^{-1})}{(1 - z^{-1})(1 - 0.0104z^{-1})}$$

Control ratio: $H(z) = 0.7912z^{-1}(1 + 0.2639z^{-1})$

Step response error: $E(z) = 1 + 0.2088z^{-1}$

Two-controller minimum sensitivity design (k = 1):

$$D_1(z) = \frac{7.2454}{(1 - 0.4944z^{-1})}, \qquad D_2(z) = \frac{1.9019(1 - 0.3389z^{-1} + 0.0034z^{-2})}{(1 + 0.2639z^{-1})}$$

Two-controller minimal sensitivity design:

$$D_1(z) = 7.2454, \qquad D_2(z) = \frac{1.2770(1 - 0.0104z^{-1})}{(1 + 0.2639z^{-1})} .$$

One-controller design:

$$D(z) = \frac{7.2454(1 - 0.0104z^{-1})}{(1 + 0.2088z^{-1})}$$

For a 20% increase of Ra

The varied plant: $$G_v(z) = \frac{0.0893z^{-1}(1 + 0.2567z^{-1})}{(1 - z^{-1})(1 - 0.0093z^{-1})}$$

Two-controller minimum sensitivity system:

$$E_v(z) = 1 + 0.3530z^{-1} + 0.0102z^{-2} - 0.0302z^{-3} - 0.0097z^{-4} + 0.0z^{-5} + 0.0010z^{-6} + 0.0006z^{-7} + 0.0001z^{-8} + \cdots$$

Two-controller minimal sensitivity system:

$$E_v(z) = 1 + 0.3530z^{-1} + 0.0685z^{-2} + 0.0130z^{-3} + 0.0025z^{-4} + 0.0005z^{-5} + \cdots$$

One-controller system:

$$E_v(z) = 1 + 0.3530z^{-1} + 0.0943z^{-2} + 0.0290z^{-3} + 0.0084z^{-4} + 0.0025z^{-5} + \cdots$$

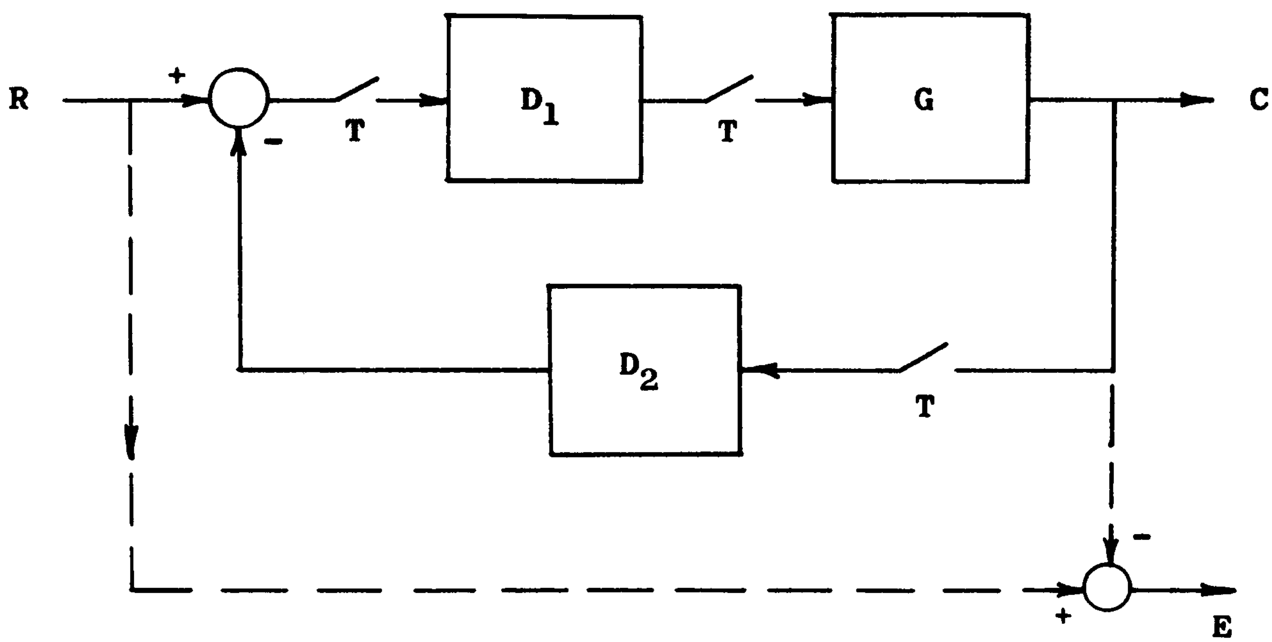

Fig. 1. Two-controller sampled-data control system.

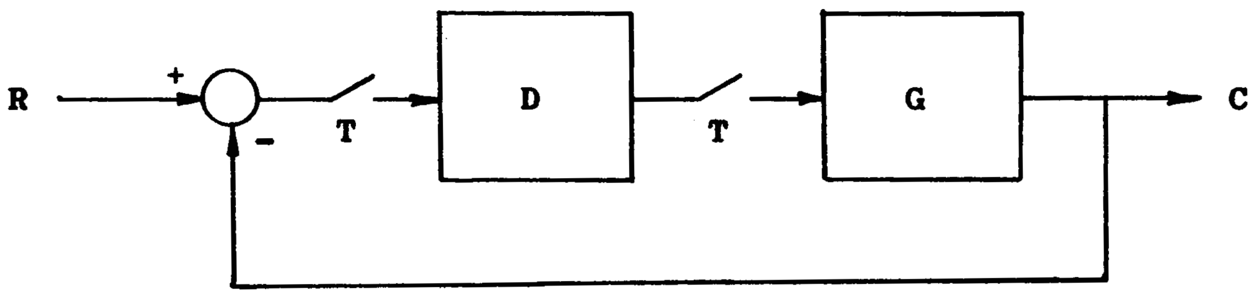

Fig. 2. One-controller sampled-data control system.

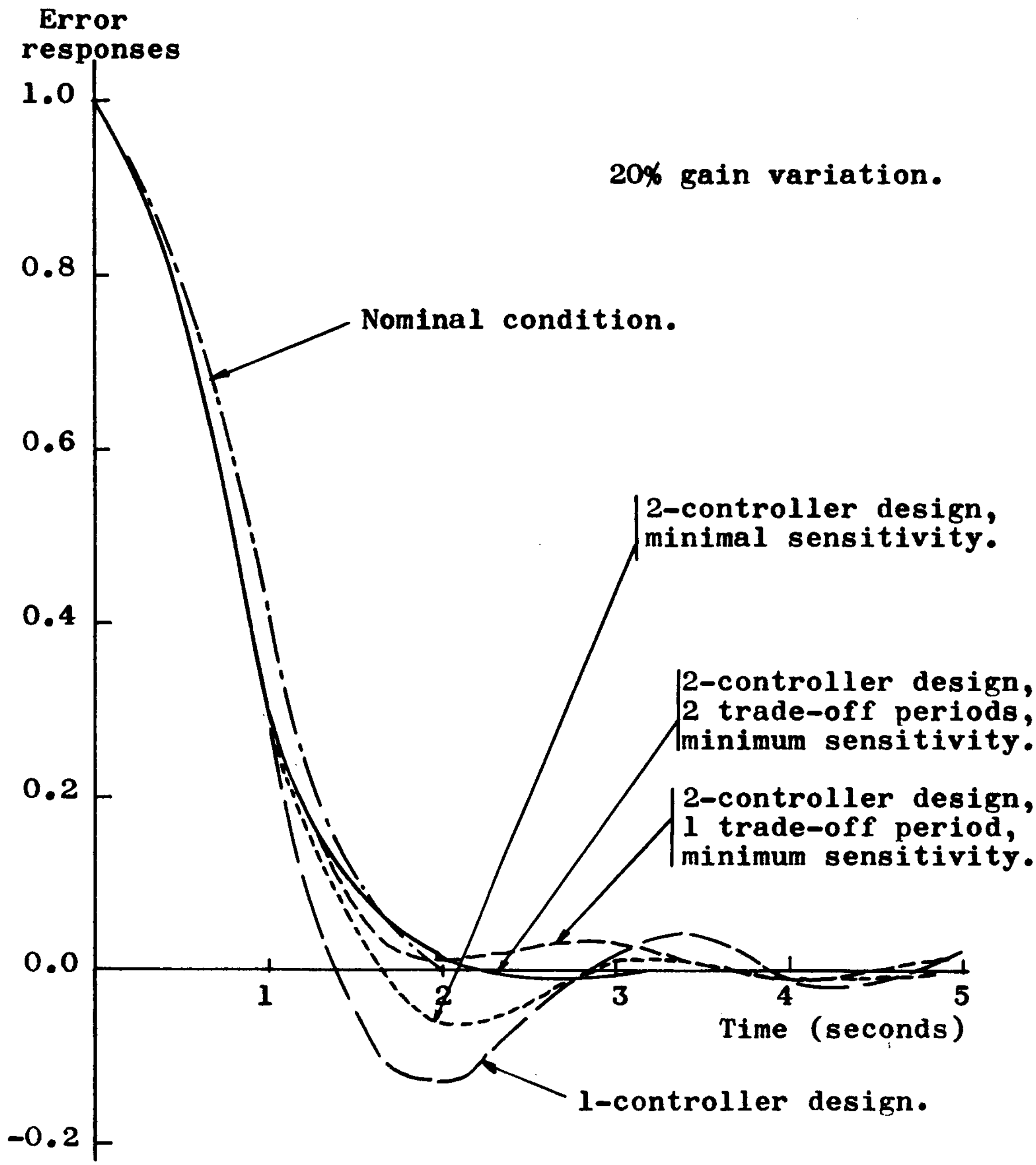

Fig. 3. System errors for Example 1.

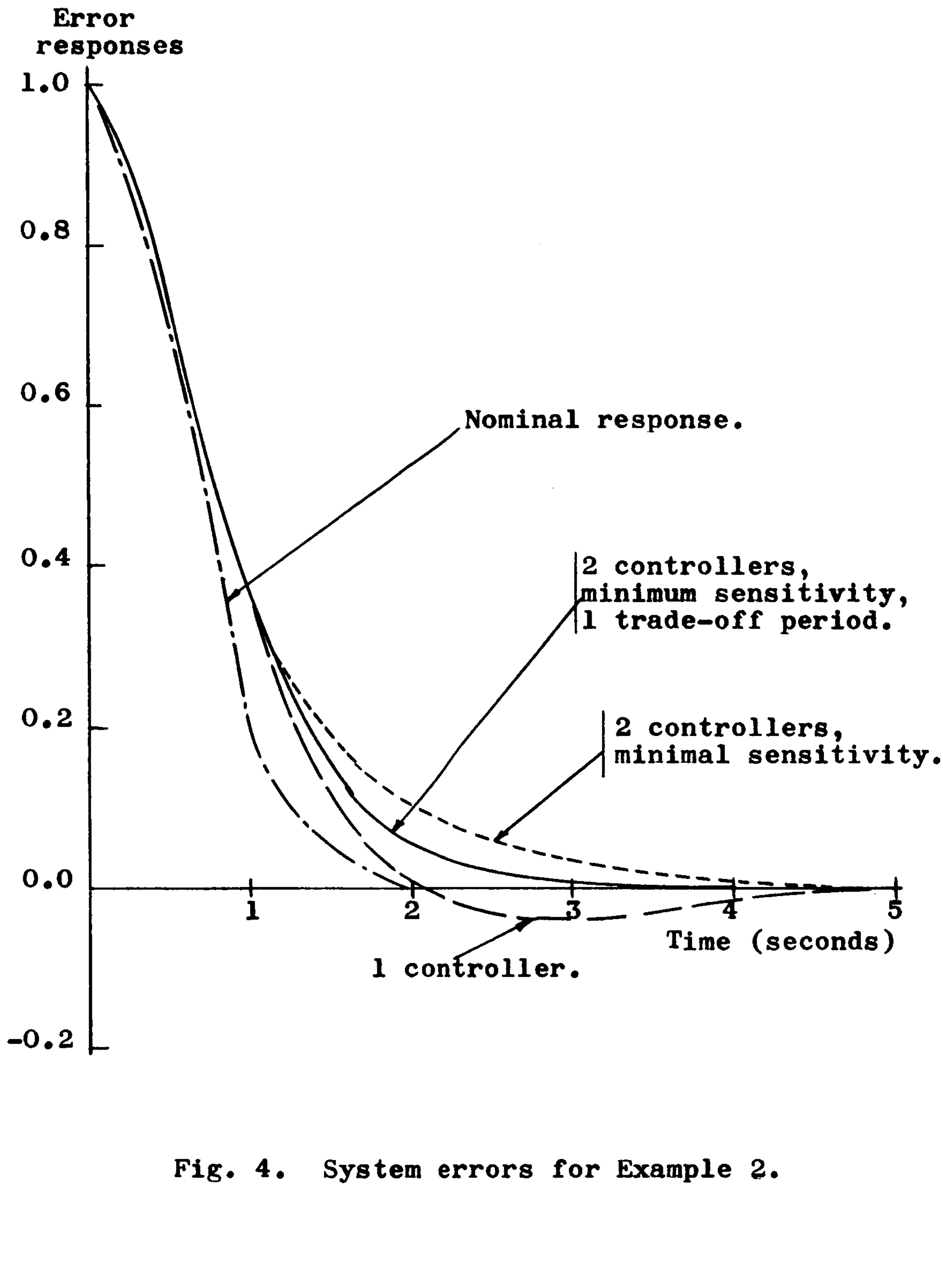

Fig. 4. System errors for Example 2.